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A STOCHASTIC SOLUTION CONCEPT FOR n -PERSON GAMES

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I. INTRODUCTION

The abundance of solution concepts in cooperative game theory is testimony both to its richness and to its elusive and seemingly indeterminate nature. One difficulty with virtually all current solution concepts is that they often fail to make a prediction or they predict indiscriminately. Thus, the universally embraced core frequently fails to exist, while von Neumann-Morgenstern solutions and bargaining sets sometimes fail to exist and often generate a multiplicity of solution sets too inclusive to allow for meaningful falsification. Recent interesting solution concepts such as the competitive solution (McKelvey, Ordeshook and Winer, 1978) and the "top cycle" (Schwartz, 1970) have similar drawbacks.

While it seems quite unreasonable to expect a single game theoretic model to generate viable solutions over a large domain of games, the fact remains that outcomes in games are invariably achieved and such outcomes are observed to cluster in significant patterns. This latter observation applies to recent committee voting experiments (Fiorina and Plott, 1978; McKelvey, Ordeshook and Winer, 1978) as well as to more classical game theory experiments (Kalisch, et al, 1952).

This paper develops a stochastic solution concept for general n-person games in characteristic function form (with or without side payments). Thus, the prediction of the model presented is a probability distribution over the space of possible outcomes (payoffs). The advantages of such an approach are twofold. First, the model always makes a probabilistic prediction which can be computed directly or approximated by simulation. Secondly, instead of requiring a "yes" or "no" answer to the question of whether a given outcome or set of outcomes is to be exalted as being in "the solution," we obtain an associated probability. It is hoped that the elusive "patterns" observed in game theoretic behavior may be explained nondeterministically. We note in passing the conceptual parallel between our approach and the considerably more complex and highly successful quantum theory of matter (the "players" are atoms rather than people).

The model we develop starts with a natural dominance relation on the outcomes and relies heavily on the theory of Markov chains.¹ The ideas evolve from and generalize an earlier model of legislative decisionmaking (Ferejohn, Fiorina and Packel, 1978). The model and its results subdivide into two cases. When the space of outcomes is finite, the theory of finite Markov chains leads to some easily proved theorems and a natural approach to computing limiting probabilities. The second situation considered allows a continuum of outcomes, imbedded in a Euclidean space R^m . We refer to this as the "spatial" case. Here the theory of continuous state Markov chains applies. Theoretical development for this case is more difficult and limiting probabilities

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have, to date, only been approachable by Monte Carlo simulation.

The rest of the paper is organized as follows. The next section presents the required preliminaries on characteristic function form games without side payments, from which the more familiar side payment games result as a special case. The two subsequent sections define the stochastic solution and develop relevant theorems for the finite and spatial cases. In each case we are able to show that the stochastic solution coincides with a restriction of the core, which we call a strong core, whenever a strong core exists. The penultimate section considers applications to simple games, side payment games, and legislative decisionmaking. Predictions of the model are computed and compared with previous experimental work. The concluding section discusses possible refinements and generalizations of the model and suggests directions for additional exploration.

II. GAMES IN GENERALIZED CHARACTERISTIC FUNCTION FORM

The standard formulation of games in characteristic function form implicitly assumes transferable utility and side payments. In what follows we use a more general formulation which does not require these assumptions. We refer the reader to Aumann and Peleg (1960), where a version of these more general ideas was first considered, and to Rapoport (1970) for motivation as to their usefulness in nonconstant sum games. An exposition of what we shall call generalized characteristic function form games, along with a case for the importance of games without side payments in

political voting processes, can also be found in McKelvey, Ordeshook and Winer (1978).

Let $N = \{1, 2, \dots, n\}$ denote the set of players and let X be a set of outcomes among which the players must choose. If X is finite we need impose no topological structure upon it. Otherwise, we assume X is a compact subset of R^m , endowed with its Euclidean topology and Lebesgue measure. If we assume that each $x \in X$ has associated with it a utility or "payoff" to each player, then there is no loss of generality in assuming X to be a subset of R^n , which we do when it is convenient.

A game in generalized characteristic function form is a function $v: 2^{N \setminus \{\phi\}} \rightarrow 2^X$. The interpretation of v is as follows: for each (nonempty) coalition C of players, $v(C)$ denotes the set of all outcomes that C can guarantee its members regardless of actions by players outside of C . If there are no outcomes that a coalition C can ensure itself, then $v(C) = \phi$.

It is natural and consistent to regard v as a simple game when each coalition C has either $v(C) = X$ or $v(C) = \phi$. The collection of winning coalitions could then be defined by $W = \{C \subseteq N \mid v(C) = X\}$. If $X \subseteq R^m$, transferable utility is assumed, and side payments are allowed, each coalition C would clearly be wise to maximize its joint payoff over $v(C)$. In this case the above formulation reduces to a standard game in characteristic function form $(v: 2^{N \setminus \{\phi\}} \rightarrow R)$. These special cases will be revisited in Section V.

To proceed with our development, we require for each player $k \in N$ an asymmetric binary relation $>_k$ on X which gives k 's preferences over the outcomes in X . Note that such an ordering results naturally if we assume that the elements of X are monetary payoffs. In this case we would define, for $x, y \in X \subseteq \mathbb{R}^m$, $y >_k x \iff y_k > x_k$ (where the k^{th} coordinate x_k of x denotes the payoff to player k associated with outcome x). The collection of individual preferences induces, for each coalition C , a coalitional preference $>_C$ on X defined by

$$y >_C x \iff y >_k x \forall k \in C.$$

We say that y is directly accessible from x via C , written $x \xrightarrow{C} y$, if C prefers y to x and has the power to enforce the outcome y .²

Formally,

$$x \xrightarrow{C} y \iff y >_C x \text{ and } y \in v(C).$$

To get a measure of the degree of direct accessibility of y from x , we define

$$c(x, y) = |\{C \subseteq N \mid x \xrightarrow{C} y \text{ and } \sim(x \xrightarrow{C'} y) \forall C' \subseteq C\}|.$$

Thus $c(x, y)$ is the number of minimal coalitions via which y is directly accessible from x . We are now ready to proceed with the development of our stochastic solution concept for a game v each of whose players has a preference ordering over the outcomes X .

III. THE STOCHASTIC SOLUTION IN THE FINITE OUTCOME CASE

Given a game $v: 2^N \setminus \{\emptyset\} \rightarrow 2^X$ with $|X| = m$, we let $X = \{1, 2, \dots, m\}$ for convenience.³ The model motivating our stochastic solution concept is based upon the following assumptions:

- (1) A starting (or default) outcome is selected stochastically. (In the absence of a more natural choice, we might allow each outcome to be the starting state with probability $1/m$, though in "most" cases our results will be seen to be independent of the starting state.)
- (2) At each step the currently held outcome (a temporary status quo) is considered in pairwise opposition to each other outcome.
- (3) The probability at each step of moving from a currently held outcome i to an outcome j is proportional to $c(i, j)$.
- (4) The process is "equally likely" to stop (and to choose the currently held outcome) after any number $r = 0, 1, 2, \dots$ of steps.

While it is clear that very few voting or game procedures satisfy these assumptions to the letter, there is some evidence that voters and experimental subjects often tend to operate on alternatives in pairwise fashion. Assumption 3 is at the foundation of our models and may not (in the presence of assumption 4) be unreasonable, especially when N and X are large enough to impede systematic enumeration of preferred outcomes and their supporting coalitions. Assumption 4 is clearly idealized, and may not be necessary if "convergence" is rapid enough. Also, the model may be easily adapted to other probability distributions on the number of steps required for termination. Finally, even if the assumptions on which the model is based can in no way be seen to operate, it is hoped that the model will still generally serve as a reasonable

probabilistic predictor of chosen outcomes. Clearly this needs to be tested experimentally.

For each $i, j \in X$, $1 \leq i, j \leq m$, let $c_{ij} = c(i, j)$. We now use the nonnegative integers c_{ij} to set up a stochastic transition matrix $P = (p_{ij})$ as follows:

$$p_{ij} = \begin{cases} c_{ij} / \sum_{k=1}^m c_{ik} & \text{if } \sum_{k=1}^m c_{ik} > 0 \\ \delta_{ij} & \text{otherwise} \end{cases}$$

(where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$). For any nonnegative integer r , the product matrix P^r is also stochastic and its (i, j) entry is the transition probability of going in exactly r steps to outcome j , given that i is the current outcome. The solution concept we seek now evolves directly from the theory of finite Markov chains.

Assumption 4 suggests looking at $P^{(r)} = \frac{1}{1+r} \sum_{k=0}^r P^k$ and $P^{(\infty)} = \lim_{r \rightarrow \infty} P^{(r)}$, prime objects of interest in the Markov theory. The matrix $P^{(r)}$ gives transition probabilities in r or fewer steps and it is a standard result that $P^{(\infty)}$ always exists and is a stochastic matrix (see, for instance, Kemeny and Snell, 1960).

Let $Q(0) = (q_1, q_2, \dots, q_m)$ be a row vector whose j^{th} entry is the probability that the starting outcome is j . Then $Q = Q(0)P^{(\infty)}$ provides the desired limiting probability distribution on the outcomes and serves to define the stochastic solution to the game v . While it is not inconceivable that a solution concept should depend on the starting or default outcome, there exists a natural necessary and sufficient condition under which the stochastic solution Q will be independent of $Q(0)$. As expected, the results depend

squarely upon the elementary theory of finite Markov chains, which we now develop.

Returning for this paragraph to the case of a general outcome space X , we say that outcome y is (not necessarily directly) accessible from outcome x (written $x \rightarrow y$) if either $x = y$ or there exists $z_0, z_1, \dots, z_t \in X$ with $z_0 = x$, $z_t = y$, and $z_k \xrightarrow{C_k} z_{k+1} \forall k = 0, 1, \dots, t-1$ (accessibility is the transitive closure of direct accessibility). If both $x \rightarrow y$ and $y \rightarrow x$, we say x and y communicate. It is immediate that communication is an equivalence relation, decomposing X into disjoint sets which we call communication sets. A communication set C is ergodic if $x \in C$, $y \notin C \Rightarrow \sim (x \rightarrow y)$. (When X is finite this is equivalent to $i \in C$, $j \notin C \Rightarrow$ the (i, j) entry of P^r is 0 for all r). If all outcomes communicate, so that X itself is a communication (and trivially an ergodic) set, the matrix P and the game v giving rise to it are called irreducible. More generally, we call P and v singly ergodic if there is precisely one ergodic set in X .⁴

Theorem 1: Let v be a game over a finite set of outcomes. Then the stochastic solution Q for v is independent of the starting distribution $Q(0)$ if and only if v is singly ergodic.

Proof: See Doob (1953), pp. 170-181.

q.e.d.

We note that Q is independent of the starting distribution if and only if all rows of $P^{(\infty)}$ are identical. In this case, Q is the unique probability vector solution to the equation $QP = Q$. The extent to which the stochastic solution depends upon $Q(0)$ can be

estimated by considering the probability that a "randomly generated" direct accessibility relation yields an irreducible matrix. Such results are described in Moon (1968), where it is seen that the probability of irreducibility rapidly approaches unity as the number of players increases beyond 10.⁵

The standard game theoretic notion of a core for games characterized by a dominance relation singles out the set of undominated outcomes (these singleton ergodic sets are called absorbing states in the Markov theory). In our formulation, the core E is the union of the singleton ergodic sets and can be defined directly by

$$E = \{x \in X \mid \forall y \in X \ni x \rightarrow y\}.$$

It is easy to show (see Section V, Example 1) that this core condition is not strong enough to ensure that E , when it exists, will always be an event of probability one according to the stochastic solution. It seems natural to define a more restrictive concept, the strong core E_s , by

$$E_s = \{x \in E \mid \forall y \in X \text{ with } y \neq x, y \rightarrow x\}.$$

Thus, outcomes in the strong core are both undominated and accessible from all other outcomes, and it follows that E_s is a singleton whenever it is nonempty. It can also be checked that $E = E_s$ if and only if v is singly ergodic and that $E = E_s$ whenever v is irreducible.

When X is finite, $A \subseteq X$, and $Q = (q_1, q_2, \dots, q_m)$ is a probability distribution on X , we let $Q(A) = \sum_{i \in A} q_i$. We may now establish straightforward connections between the stochastic solution and the (strong) core.

Theorem 2: Let v be a game with finite outcome space X and stochastic solution Q (which may depend upon $Q(0)$).

- i) $E_s \neq \emptyset \Rightarrow Q(E_s) = 1$.
- ii) $Q(E) = 1 \vee Q(0) \Leftrightarrow$ the only ergodic sets are singletons.
- iii) If D denotes the union of all ergodic sets, then $Q(D) = 1 \vee Q(0)$.

Proof: i) Let $E_s = \{j\}$. Since j is undominated, it follows that the j^{th} row of any power of the matrix P is 1 on the diagonal and 0 elsewhere. Since j is accessible from all other alternatives, there must exist some positive integer s such that all entries in the j^{th} column of P^s are positive. Thus every outcome has a positive probability of leading to j after exactly s steps. Let $p > 0$ be the minimum of these m probabilities. Then the probability of failing to get "stuck" at j from any i after ks steps is $\leq (1-p)^k$. Since $(1-p)^k \rightarrow 0$ as $k \rightarrow \infty$, the probability of getting to j after r steps approaches unity as $r \rightarrow \infty$. We conclude that $p(E_s) = 1$.

ii) \Rightarrow : An ergodic set C with two or more members must by definition be disjoint from E and hence any $Q(0)$ which is non-zero on C must result in $Q(C) > 0$ and hence $Q(E) < 1$.

\Leftarrow : By reasoning similar to that of i), the ergodic sets contain the only outcomes on which Q can have positive probability. Since E is the union of the singleton ergodic sets, we must have $Q(E) = 1$ regardless of $Q(0)$.

iii) This follows by an obvious extension of the arguments for i) and ii).

q.e.d.

This completes our treatment of the finite outcome case. Note that for relatively small numbers of outcomes the matrix $P^{(\infty)}$ and hence the solution Q can be determined algebraically or approximated computationally by matrix methods. For large finite outcome spaces (such as a lattice point approximation of a spatial game) Monte Carlo simulation appears to provide stable qualitative results in a relatively small number of steps.

IV. THE STOCHASTIC SOLUTION IN THE SPATIAL OUTCOME CASE

In this section we assume that X is a compact subset of \mathbb{R}^m (Euclidean topology) and we let μ denote Lebesgue measure on \mathbb{R}^m . To ensure the convergence needed for the existence of the stochastic solution, the individual preferences over X will have to satisfy certain conditions. Further restrictions on preferences will be needed to get "core preserving" results analogous to Theorem 2. Granting such restrictions on preferences, virtually all of the finite outcome developments have natural spatial analogs. Likewise, the assumptions which motivated the finite outcome model carry over if discrete probability distributions are replaced by probability measures or associated probability distribution functions. As expected, the added technicalities called for in the spatial case closely parallel the analytic complexities that arise in going from finite state to continuous state Markov chains. We refer the reader to Doob (1953) for expository details.

For each $x \in X$, let $a(x) = \{y \in X \mid x \xrightarrow{C} y \text{ for some } C \subseteq N\}$, the set of outcomes directly accessible from x . It is immediate

that $c(x, y) > 0$ if and only if $y \in a(x)$. This definition provides an alternative means of defining the core E , since $E = \{x \in X \mid a(x) = \emptyset\}$. Symmetrically, we define $b(x) = \{y \in X \mid y \xrightarrow{C} x \text{ for some } C \subseteq N\}$, the set of outcomes from which x is directly accessible. We will invoke the following assumptions:

A1: $\forall x \in X$, $a(x)$ and $b(x)$ are open in X .

A2: $E = \emptyset \Rightarrow \exists \epsilon > 0 \exists \mu(a(x)) \geq \epsilon \forall x \in X$.

A3: If $a(y_0) = \emptyset$, \exists a base \mathcal{B} of open sets at y_0 such that $(\forall B \in \mathcal{B}, y \in B, x \notin B), x \leftrightarrow y$ and $\sim(y \leftrightarrow x)$.

Whether or not these assumptions are satisfied ultimately depends upon the nature of the individual preferences on X . Thus, if for each $x \in X$ and each player $\rho \in N$ the preference $>_\rho$ forces $\{y \mid y >_\rho x\}$ to be open in X , then A1 will hold since unions of finite intersections of open sets are open. A compactness argument can be used to show that A1 and A2 will both hold if each individual preference $>_\rho$ can be represented by a continuous (utility) function $U_\rho: X \rightarrow \mathbb{R}$ in the sense that $y >_\rho x \iff U_\rho(y) > U_\rho(x)$. A similar statement applies if, for each $\rho \in N$, the correspondence $x \mapsto \{y \in X \mid \sim(y >_\rho x)\}$ is upper and lower semicontinuous on X (for definitions see Debreu, 1959). Assumption A3 is more delicate and seems to call for both continuity and "thin" indifference requirements on individual preferences. This assumption is a spatial version of the strong core definition given in the previous sections. It will be used both to establish the existence of a limit distribution and to identify it with the core in the case where the core is nonempty.⁶

Assuming A1, A2 and A3 in what follows, we now proceed with the continuous state Markov chain development. For each $x \in X$ and each Lebesgue measurable subset B of X , define

$$p(x, B) = \begin{cases} \int_B c(x, y) dy / \int_X c(x, y) dy & \text{if } a(x) \neq \emptyset \\ \chi_B(x) & \text{if } a(x) = \emptyset \end{cases}$$

where integration is with respect to Lebesgue measure on $X \subseteq \mathbb{R}^m$ and χ_B is the characteristic function of B , defined by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

For each $x \in X$, the function $p(x, \cdot)$ is then a probability measure on the space of Lebesgue measurable subsets of X , and $p(x, B)$ represents the probability of going in a single step into the subset B given that x is the current outcome. The collection of all these measures as x ranges over X forms a continuous state Markov chain (the continuous analog of our stochastic matrix) whose "discrete-time" behavior we wish to investigate. The r -step transition probabilities evolve inductively as follows:

$$p^0(x, B) = \chi_B(x)$$

$$p^1(x, B) = p(x, B)$$

$$p^{r+1}(x, B) = \int_X p(y, B) p^r(x, dy),$$

where $p^r(x, dy)$ denotes integration with respect to y using the measure $p^r(x, \cdot)$.

$$\text{We now look at } p^{(r)} = \frac{1}{r+1} \sum_{k=0}^r p^k \text{ and } p^{(\infty)} = \lim_{r \rightarrow \infty} p^{(r)}.$$

In the following theorem we prove the existence of the limit $p^{(\infty)}$.

Theorem 3: Given a spatial game v with associated Markov chain p and satisfying A1, A2 and A3, the limiting distribution $p^{(\infty)}(x, \cdot)$ exists for every $x \in X$.

Proof: Case I, $E = \emptyset$: In this case $a(x) \neq \emptyset \forall x \in X$ and $p(x, B)$ can be defined in terms of the density function $p_0(x, y) = c(x, y) / \int_X c(x, y) dy$

since $p(x, B) = \int_B p_0(x, y) dy$. By results from Doob (1953, pp. 192-

214), it suffices to show that $\mu(X) < \infty$ and that p_0 is bounded on $X \times X$. The first fact is immediate since X is compact. The boundedness of p_0 results by noting (conservatively) that $c(x, y) \leq 2^n \forall (x, y) \in X \times X$ and $c(x, y) \geq 1 \forall y \in a(x)$. We then have, using A2,

$$p_0(x, y) = c(x, y) / \int_X c(x, y) dy \leq 2^n / \mu(a(x)) \leq 2^n / \epsilon \quad \forall (x, y) \in X \times X.$$

Case II, $E \neq \emptyset$: From A3 it is immediate that E must contain exactly one outcome, which we denote by y_0 . Given $B \in \mathcal{B}$ and $x \notin B$, A1, A3, and the definition of $x \mapsto y$ can be used to obtain an open set U_x containing x and a $y_x \in B$ such that y_x is accessible in a fixed number s_x of steps from every member of U_x . The sets $\{U_x\}_{x \notin B}$ form an open cover of the compact set $X \setminus B$ and hence give rise to a finite subcover $\{U_{x_t}\}_{t=1}^T$. Letting $s = \max_{1 \leq t \leq T} \{s_{x_t}\}$, we conclude from the form of $p(x, B)$ (and A1) that there is $q > 0$ such that

$p^S(x, B) \geq q \forall x \in X \setminus B$. By A3, the process is "stuck" in B once it gets there. Now we proceed in a fashion similar to the finite outcome argument of Theorem 2. The probability of failing to enter B from any $x \in X \setminus B$ after ks steps is $\leq (1-q)^k$. Since $(1-q)^k \rightarrow 0$ as $k \rightarrow \infty$, the probability of being (and staying) in B after r steps approaches unity as $r \rightarrow \infty$. Thus $p^{(\infty)}(x, B) = 1 \forall x \in X \setminus B$. Since $B \in \mathcal{B}$ was arbitrary and \mathcal{B} is an open base at y_0 , we conclude that $p^{(\infty)}(x, \cdot)$ exists for all $x \in X$ and that

$$p^{(\infty)}(x, B) = \begin{cases} 1 & \text{if } y_0 \in B \\ 0 & \text{otherwise} \end{cases} . \quad \text{q.e.d.}$$

As in the finite outcome case, we let $Q(0)$ define a starting probability distribution on X . Here we take $Q(0)$ to be a density function so that, for $B \subseteq X$, $\int_B Q(0)(x) dx$ is the probability of starting with an outcome in B. The stochastic solution for the spatial case is the probability distribution Q on the Lebesgue measurable subsets of X defined by

$$Q(B) = \int_X Q(0)(x) p^{(\infty)}(x, B) dx,$$

giving the probability of a final outcome in B.

Corollary: Under the hypotheses of Theorem 3, if a core E for the game v exists, then E is a singleton set and $Q(E) = 1$.

Proof: This follows directly from the last part of the proof of Theorem 3 and the definition of Q .

q.e.d.

Conditions under which Q is independent of the starting distribution $Q(0)$ are precisely as in Theorem 1. We state the result for completeness.

Theorem 4: Given a spatial game v satisfying A1, A2 and A3. Then the stochastic solution Q is independent of the starting distribution $Q(0)$ if and only if v is singly ergodic.

Proof: See Doob (1953, pp. 192-214) for the case where $E = \emptyset$. If $E \neq \emptyset$, Theorem 3 and its corollary provide the proof. q.e.d.

It is unclear at this time just what connections might exist between A1, A2 or A3 (or continuity assumptions on individual preferences) and single ergodicity.⁷ It seems plausible that conditions like connectivity or convexity of X may also be important in this regard.

It should be clear that unless some very convenient preferences are used in the spatial case, computing solutions is likely to prove very difficult. A useful alternative, when preferences are computationally tractable, is an approach using Monte Carlo methods. This is the method used for estimating spatial results in the next section.

V. APPLICATIONS AND EXAMPLES

In this section we illustrate the stochastic solution by computing or approximating it in a variety of situations. When possible, we compare the results obtained with existing experimental data. The purpose of such comparison is not to confirm or reject

TABLE 1

EXPERIMENTAL AND SIMULATION RESULTS FOR GAME OF EXAMPLE 3

Run Number	Experimental Runs		Simulation Runs	
	Outcome	Enforcing Coalitions	Outcome	Enforcing Coalition
1	(.00,.40,.30,.30)	{2,3,4},{1}	(.30,.37,.06,.21)	{1,2}
2	(.00,.43,.43,.15)	{2,3,4},{1}	(.37,.43,.18,.02)	{2,3}
3	(.13,.38,.38,.13)	{1,4},{2,3}	(.31,.22,.18,.29)	{1,2}
4	(.13,.44,.44,.00)	{1,2,3},{4}	(.41,.31,.16,.11)	{1,2}
5	(.25,.50,.13,.13)	{1,2},{3,4}	(.06,.42,.36,.16)	{2,3,4}
6	(.43,.43,.00,.15)	{1,2,4},{3}	(.33,.06,.30,.32)	{1,3,4}
7	(.19,.44,.38,.00)	{1,2,3},{4}	(.06,.44,.21,.28)	{2,3}
8	(.44,.44,.00,.11)	{1,2,4},{3}	(.30,.44,.17,.09)	{1,2}
Mean	(.20,.43,.25,.12)		(.28,.34,.20,.18)	
Standard Deviation	(.16,.03,.17,.09) \Rightarrow .25 total	48 Runs { Combined Mean Combined Standard Deviation	(.13,.13,.09,.10) \Rightarrow .22 total (.25,.29,.26,.20) (.10,.15,.12,.13) \Rightarrow .25 total	
Shapley Value	(.25,.33,.25,.17)		(.25,.33,.25,.17)	

required to obtain a (strong) core (Plott, 1967; Davis, DeGroot, and Hinich, 1972), which occurs at x^3 .

Summarized in Figure 1 are ten experimental runs of this game (from Fiorina and Plott, 1978). Figure 2 plots simulation results of ten runs each carried out to ten steps and also gives descriptive statistics of fifty such runs. In all experimental and simulation runs, the upper right-hand corner point (200,150) was taken as the starting outcome. Qualitatively, the theoretical and experimental results appear to be in accord, though convergence as a function of the number of steps is somewhat slower in the Markov model if we regard a "step" in an experimental run as the passage of a motion giving a new status quo.¹¹

[Figures 1 and 2 Here]

Example 5: This differs from Example 4 "only" in that voter 3 has his ideal point moved from the previous core point (39,68) to the point (51,59). There is no longer a core. It can be shown that there is only one ergodic set and it properly contains the Pareto optimal quadrilateral determined by the other four ideal points.

Figures 3 and 4 summarize, respectively, experimental results (from Fiorina and Plott, 1978) and results of fifty runs of a Monte Carlo simulation where termination occurred after fifty steps.

[Figures 3 and 4 Here]

We do not offer any comparative statistical analyses of the results obtained in Examples 4 and 5, contenting ourselves with the comment that the visual results are striking.¹² Note that the Fiorina-Plott experiments used a simple set of parliamentary

Very little experimental work seems to have been done in these finite outcome situations. It would clearly be of interest for our purposes to have the results of such experiments. Even the exceedingly simple Example 1 raises the interesting question of whether real players would reach outcome 5 without having started there.⁸

The third example illustrates a spatial game in which side payments are allowed. It is a four-person constant sum game in normalized form. We take the outcome space to be the set of imputations (payoffs whose entries sum to 1) along with the starting outcome $\underline{0} = (0,0,0,0)$.

Example 3: $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_i \geq 0 \text{ and } \sum_{k=1}^4 x_k = 1\} \cup \{\underline{0}\}$,

$N = \{1,2,3,4\}$, $v(\{1,2\}) = v(\{2,3\}) = 3/4$; $v(\{1,3\}) = v(\{2,4\}) = 1/2$;

$v(\{1,4\}) = v(\{3,4\}) = 1/4$; $v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 1 & \text{if } |S| \geq 3 \end{cases}$. For each

player k , $x \succ_k y \iff x_k > y_k$.

Judiciously chosen simulation results for the stochastic solution and results of some RAND experiments (Kalisch et al, 1952 as described in Luce and Raiffa, 1957) are shown in Table 1. Each run of the simulation used a Monte Carlo approach starting at outcome $\underline{0}$ and proceeding for fifteen steps to obtain a "final" outcome. "Convergence" appeared to be rapid enough that large groups of runs of anything exceeding ten steps exhibited no qualitative differences in means or standard deviations. Both the

similarities and differences in individual runs and means are interesting. Overall means and payoff variation from run to run are similar in both experiment and theory, but coalitional behavior is somewhat different. This latter fact is not surprising given the nature of our model and the methods used in the actual experiments. In the experiments, time was limited (ten minutes) for each run and players were exhorted to act selfishly. There was neither the time nor the encouragement to consider outcomes pairwise. Also, a total of eight laboratory runs does not appear to be a usefully large sample given the variability in each run. Nevertheless, theory and experiment seem relatively compatible and both are consistent with the Shapley value for this game.⁹

[Table 1 Here]

The last two examples arise in the context of spatial modeling in legislative decision theory.¹⁰ The outcome space X is a subset of the positive quadrant in \mathbb{R}^2 and individual preferences on X are induced by locating each player (voter) at a point of X and letting the utility (preference) for a given player decrease as a continuous function of the Euclidean distance from this "ideal" point. Absolute majority rule is the voting mechanism.

Example 4: $X = [0,200] \times [0,150]$, $N = \{1,2,3,4,5\}$,

$v(S) = \begin{cases} X & \text{if } |S| \geq 3 \\ \emptyset & \text{otherwise} \end{cases}$. Figure 1 describes the ideal point x^i

for each voter $i \in N$. It is important to note that x^3 is at the intersection of the diagonals of the quadrilateral determined by the other four ideal points. This is precisely the condition

our solution concept, but simply to see to what extent and under what conditions the stochastic predictions made by our model are in reasonable accord with laboratory results. The first two examples are simple, finite alternative games using absolute majority rule as the voting procedure.

Example 1: $X = \{1, 2, 3, 4, 5\}$, $N = \{\rho_1, \rho_2, \rho_3, \rho_4\}$,

$$v(S) = \begin{cases} X & \text{if } |S| \geq 3 \\ \emptyset & \text{otherwise} \end{cases}.$$

Individual preferences on X are given by:

ρ_1	ρ_2	ρ_3	ρ_4
1	2	3	4
2	3	4	1
5	5	5	5
3	4	1	2
4	1	2	3

It should be clear by inspection that a core $E = \{5\}$ exists for this game, but that the core is not a strong core. Indeed outcome 5 is accessible from no other outcome. The transition matrix P is given by

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and there are two ergodic sets, $\{5\}$ and $\{1, 2, 3, 4\}$. In accord with Theorem 1, the stochastic solution must depend upon the starting distribution $Q(0)$. Indeed $P^{(\infty)}$ has the form

$$P^{(\infty)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and for $Q(0) = (q_1, q_2, q_3, q_4, q_5)$, we have

$$Q = \left(\frac{1-q_5}{4}, \frac{1-q_5}{4}, \frac{1-q_5}{4}, \frac{1-q_5}{4}, \frac{1-q_5}{4}, q_5 \right).$$

Example 2: $X = \{1, 2, 3, \dots, 11\}$, $N = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$,

$$v(S) = \begin{cases} X & \text{if } |S| \geq 3 \\ \emptyset & \text{otherwise} \end{cases}$$

ρ_1	1	2	8	7	3	6	10	4	5	9	11
ρ_2	2	10	3	7	4	5	6	9	1	11	8
ρ_3	3	4	7	5	6	9	11	1	8	2	10
ρ_4	4	8	10	5	9	11	1	6	2	3	7
ρ_5	5	11	1	8	10	2	6	3	7	4	9

It can be checked that this game has no core and is irreducible. Hence the stochastic solution Q is independent of $Q(0)$. In fact, computer matrix calculations yield $Q = (.16, .12, .13, .14, .15, .01, .05, .09, .03, .07, .05)$.

procedures for group decisionmaking which is close to the spirit of our status quo, pairwise comparison modeling assumptions. It is becoming increasingly clear that the choice of procedures can have immense influence upon experimental outcomes (Plott and Levine, 1978; Isaac and Plott, 1977) and the stochastic model appears to hold up well under a "sympathetic" but commonly made choice of procedures.

VI. CONCLUDING COMMENTS

Summarizing briefly what has been done here, we have proposed a stochastic solution concept for a broad class of cooperative games as the end product of a natural dynamic model of outcome selection and coalition formation. Some fundamental theory was developed for the finite and spatial outcome cases, exploiting the model's intimate connections with the theories of discrete and continuous state Markov chains. Stochastic solutions were then computed or approximated for a variety of situations and comparisons with experimental results were made when possible.

Inevitably, with a model as simple as the one proposed, a number of potentially worthwhile refinements come to mind. We summarily discuss a few of these. The requirement of minimality for a coalition to contribute to $c(x,y)$ is open to question. We have invoked minimality to avoid a form of overcounting, but it is quite conceivable that some situations might be treated better without it. In either case, the Markov chain connections and the theorems and proofs of Sections III or IV are the same. This also applies to numerous other possible ways of generating transition probabilities between alternatives.

The transition process, as currently modeled, does not rule out selection of a given outcome even though it is Pareto dominated (defeatable unanimously by all players in N) by some other outcome. Restriction to the Pareto set might make for more rapid and tighter convergence. In fact, experimental subjects do not always end up choosing Pareto optimal outcomes and our model seems to do quite well without imposing this restriction. An even more severe restriction could require each enforcing coalition to implement only outcomes in its Pareto optimal set. Such behavior would be game-theoretically natural, but might often prove to be destructively myopic in that it would maximally motivate excluded players to break up the coalition with a counter-coalition, continuing the game another step.

A refinement which might prove quite worthwhile is to graft onto the model a more sensitive stopping mechanism. While our process models the dynamics of the outcome flow very naturally, this is less true for the way the game stops. Though in most cases stochastic convergence seems rapid enough that letting the number of steps go to infinity does not pose a problem, a better stopping mechanism may be achievable by having it depend upon the number of steps elapsed, the current set of directly accessible outcomes, and the number of minimal coalitions which can enforce these outcomes. We leave these ideas for future consideration.

The computations in Table 1 for Example 3 illustrate an interesting by-product of the stochastic solution when the outcome space consists of n -tuples of payoffs to the players. By computing

FIGURE 1: Outcomes of 10 Fiorina-Plott Experiments with Core

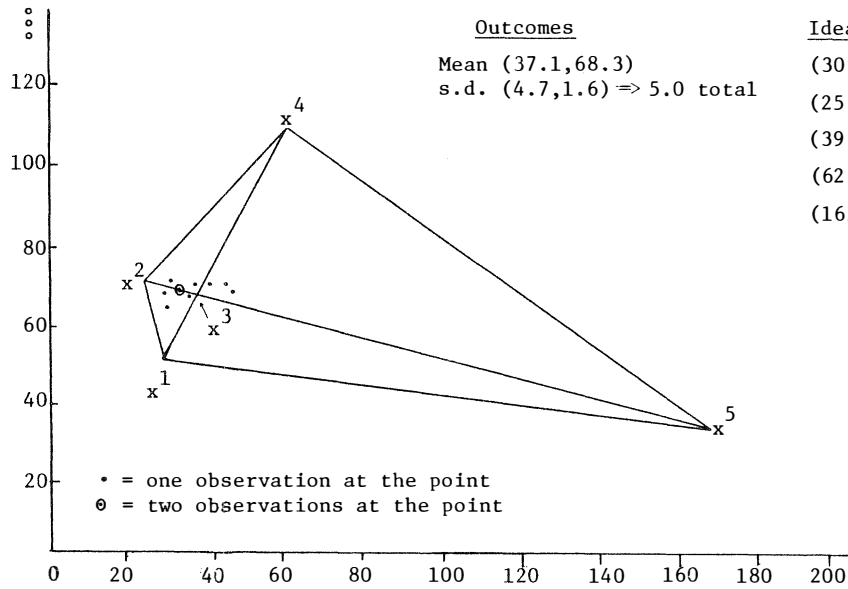


FIGURE 2: Outcomes of First 10 of 50 Simulations Runs—10 Steps Each

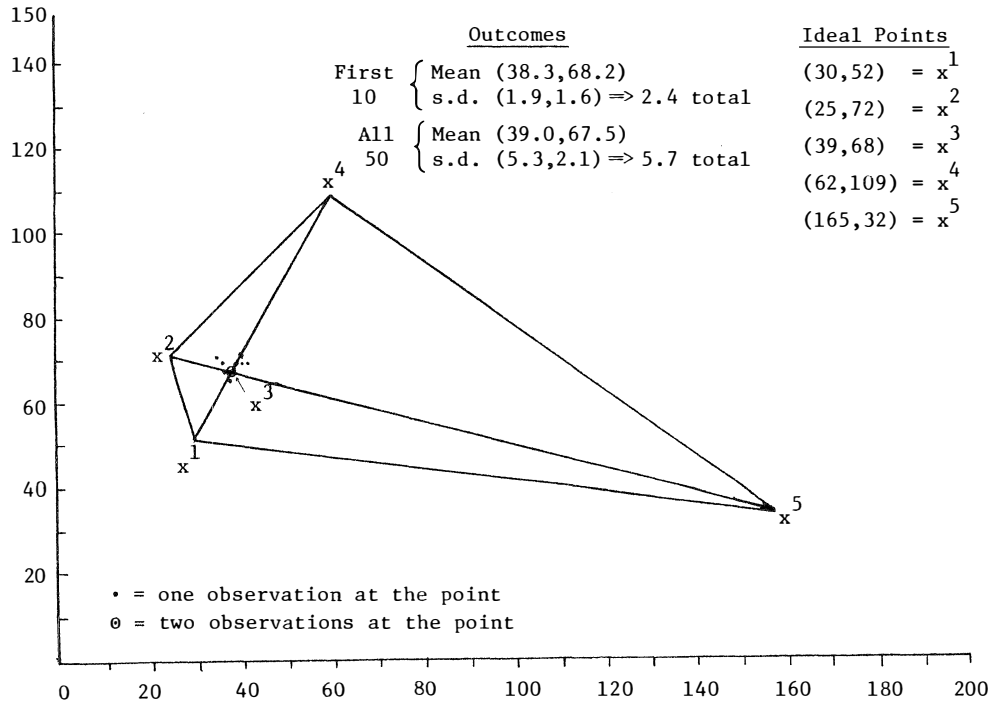


FIGURE 3: Outcomes of 15 Fiorina-Plott Experiments with No Core

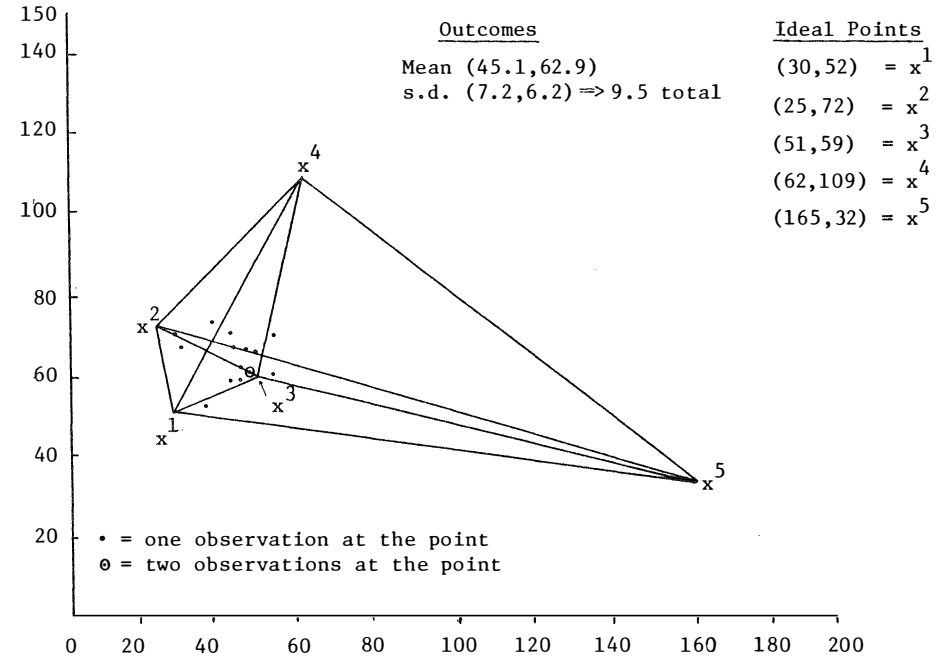
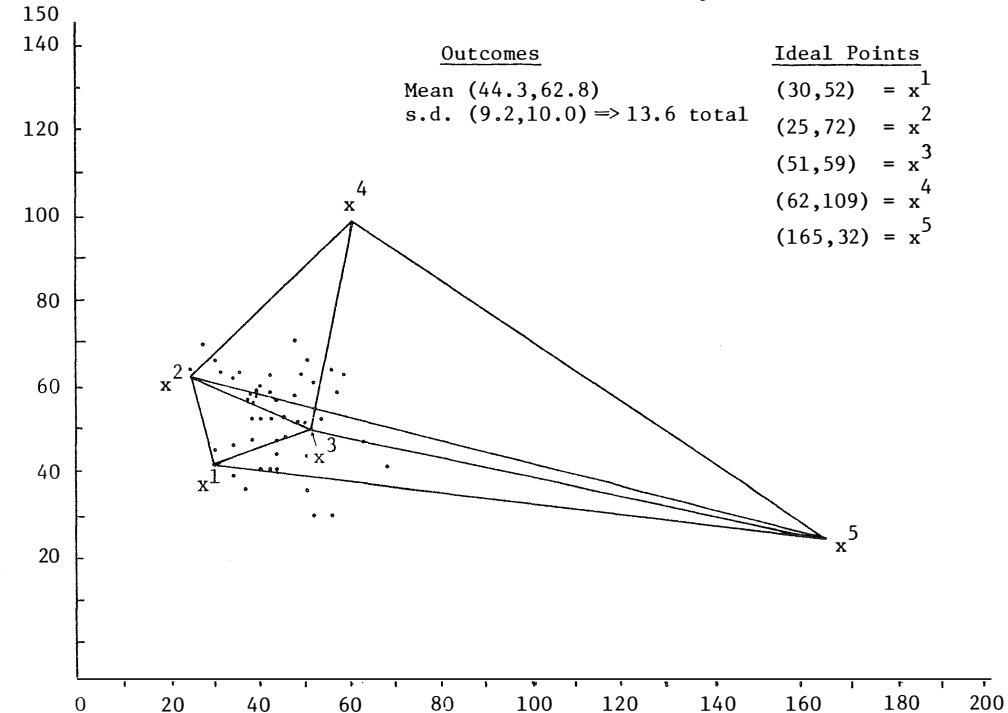


FIGURE 4: Outcomes of 50 Simulation Runs—50 Steps Each



the mean of the stochastic solution (this requires an integration in the spatial case) we obtain a "stochastic value" for characteristic function form games (see footnote 8). If V is a simple, side payment game, then this value becomes a stochastic power index.

In closing, we emphasize the need for a variety of additional carefully designed and administered experiments to test the stochastic solution and to define its areas of applicability. Our model has not assumed that the underlying game be super-additive, for instance, and it is not clear to what extent our model will be reasonable in the absence of this and other "real world" conditions. The overall idea and its currently rather sparse results seem compelling, but a vast network of operational, theoretical, experimental, and computational paths remains to be explored.

FOOTNOTES

1. Application, in this context, of the elementary theory of Markov chains is also made in Shenoy (1977). The ideas developed here arose independently and rely on more substantive aspects of the Markov theory to obtain limiting probability distributions.
2. In game theoretic terminology, the standard terminology is that " y dominates x ." We choose to employ the "accessibility" language in conformity with Markov chain terminology.
3. Confusion between outcomes and players, who have also been indexed by the positive integers will, hopefully, not arise. Henceforth the letters i and j always denote outcomes.
4. The ergodic sets are what Shenoy (1977) takes as his "elementary dynamic solutions." The "dynamic solution" is the union of all ergodic sets, so our game V will be singly ergodic precisely when the dynamic solution is an elementary dynamic solution. The idea of taking ergodic sets and their union as solution concepts appears to originate with Schwartz (1970) and has also been used by Kalai, Pazner, and Schmeidler (1976). Neither of these papers employed Markov chain ideas.

5. I am indebted to Phil Straffin for calling my attention to Moon's book and to the connections between tournaments and the ideas considered here.
6. One specific and important class of preferences for which A1 and A2 are satisfied is the Euclidean "loss function" preferences used in spatial models of legislative decision-making. With these preferences, a core "usually" does not exist (Schofield, 1977), in which case A3 holds by default. When there is a core, A3 can be shown to hold in important special cases (Ferejohn, Fiorina and Packel, 1978). Preferences which locate players on the vertices of an $n-1$ dimensional simplex in R^n and use barycentric coordinates to generate preferences would also appear to be natural and promising for the purposes of our approach.
7. In the Euclidean setting described in footnote 5, McKelvey (1976) shows that, with $X = R^m$, not only will we have single ergodicity; but, in the absence of a core, irreducibility results. An additional by-product of using this setting (and probably other continuous preference settings) is that there is no "periodicity" in the accessibility relation. As a consequence of this aperiodicity (see Dobb (1953) for details) the "Cesaro" limit $P^{(\infty)}$ can be replaced without affecting the results by $P^\infty = \lim_{r \rightarrow \infty} P^r$. A similar remark applies in the finite outcome case.

In brief, will Example 1 be a (confessedly contrived) case where the core is not likely to be the outcome of a player-symmetric simple game? Before implementing Example 1 experimentally, Charles Plott suggests adding some "losing" outcomes in a manner which will reduce the obvious "fairness" of outcome 5 by virtue of its constantly central position. This can readily be done without altering the character of the example. For useful laboratory results on Example 2, as far as testing our model is concerned, a great many runs (and a great many dollars) would be needed for comparison with our stochastic predictions.

A second experimental series of eight runs was made in the RAND experiment on a game strategically equivalent to that of Example 3. The mean of these runs was (.28, .30, .24, .18), closer both to our simulation mean and to the Shapley value. The "stochastic value" obtained by computing the mean imputation over the stochastic distribution Q is, like the Shapley value, invariant with respect to strategic equivalence.

Technically, neither the experiments nor the simulation were "spatial," but rather both used the large finite (30,000) set of outcomes determined by the integer lattice points of the set $X = [0, 200] \times [0, 150] \subseteq R^2$. It seems reasonable to expect that no significantly different results would have obtained (other than communication problems for the experi-

mental subjects) had both situations been purely spatial.

In point of fact, since all payoffs, subjects, and computers are essentially limited to a fixed finite number of decimal places, there is no such thing as a truly spatial game.

11. If we regard a "step" as the formal proposing of a new status quo, rather than actual passage, then simulation convergence is slightly more rapid than experimental "convergence." The mean number of steps under this definition in the Fiorina-Plott core experiment was sixteen. Simulation results for ten steps give roughly similar results, while such results for sixteen steps have a mean of (39.2, 67.9) and a total standard deviation of 1.1. At twenty-five steps the simulation hits the core (39,68) virtually every time. We do not attach too much significance at this point to these convergence comparisons.
12. I am indebted to Carl Lydick for his thoughtful and ingenious programming of the simulations and their graphical output for Examples 4 and 5.

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